**8.1 Matrices and Systems of Equations**

**matrix** – rectangular arrangements of object (numbers)

 $m ×n$$\left[\begin{matrix}\begin{matrix}a\_{11}&a\_{12}\\a\_{21}&a\_{22}\end{matrix}&\begin{matrix}\cdots &a\_{1n}\\\cdots &a\_{2n}\end{matrix}\\\begin{matrix}\vdots &\vdots \\a\_{m1}&a\_{m1}\end{matrix}&\begin{matrix}&\vdots \\\cdots &a\_{mn}\end{matrix}\end{matrix}\right]$

*m* = number of row

 *n* = number of columns

 a matrix is a **square matrix** if $m=n$

**Order of Matrices**

 $\left[ \begin{matrix}5&-3&0\\6&-3&10 \end{matrix}\right]$ $\left[ -44 \right]$

 $\left[\begin{matrix}\begin{matrix} 9&11\end{matrix}&\begin{matrix}-6&0 \end{matrix}\end{matrix}\right]$ $\left[ \begin{matrix}\begin{matrix}0\\13\end{matrix}&\begin{matrix}-5\\2\end{matrix}\\\begin{matrix}6\\3\end{matrix}&\begin{matrix}1\\-1\end{matrix}\end{matrix} \right]$

 $\left[ \begin{matrix}6&2&0\\-1&66&20\\28&-3&101\end{matrix} \right]$ $\left[ \begin{matrix}1\\10\\100\end{matrix} \right]$

**Matrices for a system of equations**

 System = $\left\{\begin{matrix} x + y +z=2\\2x - y =3\\ 5x-2y+ z=12\end{matrix}\right.$ coefficent matrix =

 augmented matrix =

**Elementary Row Operations**

1. Interchange two rows
2. Multiply a row by a non-zero constant
3. Add a multiple of a row to another row

Ex: interchange the first and third rows of the original matrix

 $\left[ \begin{matrix}\begin{matrix}1&3\\0&3\end{matrix}&\begin{matrix} 4&0\\-1&2\end{matrix}\\\begin{matrix}1&2\end{matrix}&\begin{matrix}-3&4\end{matrix}\end{matrix} \right]$

Ex: multiply the second row of the original matrix by $-\frac{1}{2}$

 $\left[ \begin{matrix}\begin{matrix}1& 0\\2&-8\end{matrix}&\begin{matrix}3&6\\4&0\end{matrix}\\\begin{matrix}5& 7\end{matrix}&\begin{matrix}9&4\end{matrix}\end{matrix} \right]$

Ex: add $-2$ times the first row of the original matrix to the second row

 $\left[ \begin{matrix}\begin{matrix} 1& 3\\ 2& 5\end{matrix}&\begin{matrix} 0&-1\\ 2& 3\end{matrix}\\\begin{matrix}-2& 7\end{matrix}&\begin{matrix}-3& 1\end{matrix}\end{matrix} \right]$

Ex: add $4$ times the first row of the original matrix to the third row

 $\left[ \begin{matrix}\begin{matrix} 1& 5\\ 0& 5\end{matrix}&\begin{matrix} 9&-1\\ 2& 3\end{matrix}\\\begin{matrix}-4& 7\end{matrix}&\begin{matrix}-3& 1\end{matrix}\end{matrix} \right]$

Ex: add $-7$ times the second row of the original matrix to the third row

 $\left[ \begin{matrix}\begin{matrix}1& 5\\0& 1\end{matrix}&\begin{matrix} 9&-1\\ 4& 5\end{matrix}\\\begin{matrix}0& 7\end{matrix}&\begin{matrix}-2& 1\end{matrix}\end{matrix} \right]$

**Row-Echelon Form and Reduced Row-Echelon Form**

A matrix in **row-echelon form** has the following properties

1. Any rows consisting entirely of zeros occur at the bottom of the matrix
2. For each row that doesn’t consist entirely of zeros, the first non-zero entry is a 1 (the leading 1)
3. For two successive (non-zero) rows, the leading 1 in the higher row is further to the left of the leading 1 in the lower row

A matrix is in **reduced row-echelon form** if every column with a leading 1 has zeros in every position above and below it.

Examples of *row-echelon form*:

 $\left[ \begin{matrix}\begin{matrix}1&2\\0&1\end{matrix}&\begin{matrix}-6&5\\ 3&1\end{matrix}\\\begin{matrix}0&0\end{matrix}&\begin{matrix} 1&7\end{matrix}\end{matrix} \right]$ $\left[ \begin{matrix}\begin{matrix}\begin{matrix}1&6&\begin{matrix}5&3& 2\end{matrix}\end{matrix}\\\begin{matrix}0&0&\begin{matrix}1&4& 7\end{matrix}\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&0&\begin{matrix}0&1&-1\end{matrix}\end{matrix}\\\begin{matrix}0&0&\begin{matrix}0&0& 1\end{matrix}\end{matrix}\end{matrix}\end{matrix} \right]$ $\left[ \begin{matrix}\begin{matrix}1&0\\0&0\end{matrix}&\begin{matrix}0& 3\\ 1&-6\end{matrix}\\\begin{matrix}0&0\end{matrix}&\begin{matrix}0& 0\end{matrix}\end{matrix} \right]$

Examples of *reduced row-echelon form*:

 $\left[ \begin{matrix}\begin{matrix}\begin{matrix}1&0&\begin{matrix}0&0& 7\end{matrix}\end{matrix}\\\begin{matrix}0&1&\begin{matrix}0&0& 4\end{matrix}\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&0&\begin{matrix}1&0& 3\end{matrix}\end{matrix}\\\begin{matrix}0&0&\begin{matrix}0&1&-2\end{matrix}\end{matrix}\end{matrix}\end{matrix} \right]$ $\left[ \begin{matrix}\begin{matrix}1&0\\0&1\end{matrix}&\begin{matrix}0&5\\0&1\end{matrix}\\\begin{matrix}0&0\end{matrix}&\begin{matrix}1&7\end{matrix}\end{matrix} \right]$

**Gaussian Elimination and Back Substitution**

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form and use back substitution to find the solutions for each variable.

**Solving a System of Equations using Matrices**

 $\left\{\begin{matrix}-x+ y= 4\\ 2x-4y=-34\end{matrix}\right.$ $\left[\begin{matrix}&&\\&&\\&&\end{matrix} \right]$

 (Back Substitution) $\left[\begin{matrix}&&\\&&\\&&\end{matrix} \right]$

 $\left[\begin{matrix}&&\\&&\\&&\end{matrix} \right]$

 $\left[\begin{matrix}&&\\&&\\&&\end{matrix} \right]$

**Solving a System of Equations using Matrices**

 $\left\{\begin{matrix} x+ 3y+4z= 7\\2x+ 7y+5z=10\\3x+10y+4z=27\end{matrix}\right.$ $\left[\begin{matrix}&&\\&&\\&&\end{matrix} \right]$

 (Back Substitution) $\left[\begin{matrix}&&\\&&\\&&\end{matrix} \right]$

 $\left[\begin{matrix}&&\\&&\\&&\end{matrix} \right]$

 $\left[\begin{matrix}&&\\&&\\&&\end{matrix} \right]$

 $\left[\begin{matrix}&&\\&&\\&&\end{matrix} \right]$

**Example: (no solution)**

$\left\{\begin{matrix}\begin{matrix} x+ y- z= 7\\ x +z= 2\end{matrix}\\\begin{matrix}2x+4y-z=10\\ x-y+4z=-5\end{matrix}\end{matrix}\right.$ $\begin{matrix}\begin{matrix}\\\end{matrix}\\\begin{matrix}\\\end{matrix}\end{matrix}$ $\left[ \begin{matrix}\begin{matrix}\begin{matrix}\begin{matrix}1& 1\end{matrix}&\begin{matrix}-1& 7\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}1& 0\end{matrix}&\begin{matrix} 1& 2\end{matrix}\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}\begin{matrix}2& 4\end{matrix}&\begin{matrix}-1&10\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}1&-1\end{matrix}&\begin{matrix} 4&-5\end{matrix}\end{matrix}\end{matrix}\end{matrix} \right]$

 $\begin{matrix}\begin{matrix}\\ -R\_{1}+R\_{2}\end{matrix}\\\begin{matrix}-2R\_{1}+R\_{3}\\ -R\_{1}+R\_{4}\end{matrix}\end{matrix}$ $\left[ \begin{matrix}\begin{matrix}\begin{matrix}\begin{matrix}1& 1\end{matrix}&\begin{matrix}-1& 7\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&-1\end{matrix}&\begin{matrix} 2& -5\end{matrix}\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}\begin{matrix}0& 2\end{matrix}&\begin{matrix} 1& -4\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&-2\end{matrix}&\begin{matrix} 5&-12\end{matrix}\end{matrix}\end{matrix}\end{matrix} \right]$

 $\begin{matrix}\begin{matrix}\\ -R\_{2}\end{matrix}\\\begin{matrix}\\ \end{matrix}\end{matrix}$ $\left[ \begin{matrix}\begin{matrix}\begin{matrix}\begin{matrix}1& 1\end{matrix}&\begin{matrix}-1& 7\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0& 1\end{matrix}&\begin{matrix}-2& 5\end{matrix}\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}\begin{matrix}0& 2\end{matrix}&\begin{matrix} 1& -4\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&-2\end{matrix}&\begin{matrix} 5&-12\end{matrix}\end{matrix}\end{matrix}\end{matrix} \right]$

 $\begin{matrix}\begin{matrix}\\ \end{matrix}\\\begin{matrix}-2R\_{2}+R\_{3}\\ 2R\_{2}+R\_{4}\end{matrix}\end{matrix}$ $\left[ \begin{matrix}\begin{matrix}\begin{matrix}\begin{matrix}1&1\end{matrix}&\begin{matrix}-1& 7\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&1\end{matrix}&\begin{matrix}-2& 5\end{matrix}\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}\begin{matrix}0&0\end{matrix}&\begin{matrix} 5& 6\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&0\end{matrix}&\begin{matrix} 1&-2\end{matrix}\end{matrix}\end{matrix}\end{matrix} \right]$

 $\begin{matrix}\begin{matrix}\\ \end{matrix}\\\begin{matrix}R\_{4}\\R\_{3}\end{matrix}\end{matrix}$ $\left[ \begin{matrix}\begin{matrix}\begin{matrix}\begin{matrix}1&1\end{matrix}&\begin{matrix}-1& 7\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&1\end{matrix}&\begin{matrix}-2& 5\end{matrix}\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}\begin{matrix}0&0\end{matrix}&\begin{matrix} 1&-2\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&0\end{matrix}&\begin{matrix} 5& 6\end{matrix}\end{matrix}\end{matrix}\end{matrix} \right]$

 $\begin{matrix}\begin{matrix}\\ \end{matrix}\\\begin{matrix}\\-5R\_{3}+R\_{4}\end{matrix}\end{matrix}$ $\left[ \begin{matrix}\begin{matrix}\begin{matrix}\begin{matrix}1&1\end{matrix}&\begin{matrix}-1& 7\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&1\end{matrix}&\begin{matrix}-2& 5\end{matrix}\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}\begin{matrix}0&0\end{matrix}&\begin{matrix} 1&-2\end{matrix}\end{matrix}\\\begin{matrix}\begin{matrix}0&0\end{matrix}&\begin{matrix} 0& 16\end{matrix}\end{matrix}\end{matrix}\end{matrix} \right]$

$\left\{\begin{matrix}\begin{matrix} x+ y- z=7\\ y-2z=5\end{matrix}\\\begin{matrix} z=-2\\ 0=16\end{matrix}\end{matrix}\right.$

 not possible $0\ne 16$

 no solution

**Gauss – Jordan Elimination** reduced row-echelon form

 ($2×2$ matrices only)

 $\left\{\begin{matrix}5x+3y= -7\\ x-4y= -6\end{matrix}\right.$

 $\left[\begin{matrix}&&\\&&\end{matrix}\right]$

 $\left[\begin{matrix}&&\\&&\end{matrix}\right]$

 $\left[\begin{matrix}&&\\&&\end{matrix}\right]$

 $\left[\begin{matrix}&&\\&&\end{matrix}\right]$

 $\left[\begin{matrix}&&\\&&\end{matrix}\right]$

**With a calculator**

 $\left\{\begin{matrix} x+2y=0\\ x+ y=6\\3x-2y=8\end{matrix}\right.$

 $\left\{\begin{matrix} x+y+4z=5\\2x+y - z=9\end{matrix}\right.$

 $\left\{\begin{matrix}5x+10y=25\\ x+ 2y = 5\end{matrix}\right.$

 $\left\{\begin{matrix} 5x+2y=6\\-15x-6y=0\end{matrix}\right.$